Global Journal of Science Frontier Research
Mathematics & Decision Sciences
Volume 12 Issue 2 Version 1.0 February 2012
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals Inc. (USA)
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On Quivers and Incidence Algebras

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AMS Subject Classication: *16S50.*

ON QUIVERS AND INCIDENCE ALGEBRAS

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Viji M.[α] & R.S.Chakravarti[σ]

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I. Introduction

A *quiver* ([2]) $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two set: Q_0(whose elements are called *points*, or *vertices*) and Q_1 (whose elements are called *arrows*) and two maps $s, t : Q_1 \to Q_0$ which associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Hereafter we use the notation $Q = (Q_0, Q_1)$ or simply Q to denote a quiver. A *path of length* l in Q is a sequence of arrows $(\alpha_1, \alpha_2, ..., \alpha_l)$ of Q, of length l, such that $s(\alpha_{i+1}) = t(\alpha_i)$. A path of length 0, from a point a to a is denoted by ε_a and it is called *stationary path*.

Let Q be a quiver. The *Path Algebra* KQ, of Q is the $K-$algebra, whose underlying $K-$vector space has as a basis, the set of all paths $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ of length ≥ 0. The product of 2 basis elements $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ and $(c|\beta_1, \beta_2, ..., \beta_m|d)$ of KQ is defined as,

$$(a|\alpha_1, \alpha_2, ..., \alpha_l|b).(c|\beta_1, \beta_2, ..., \beta_m|d) = \delta_{bc}(a|\alpha_1, ..., \alpha_l, \beta_1, ...\beta_m|d).$$

Let KQ_l be the subspace of KQ generated by the set Q_l of all paths of length l, where $l \geq 0$. It is clear that $(KQ_n).(KQ_m) \subseteq (KQ_{n+m})$ and we have the direct sum decomposition

$$KQ = KQ_0 \oplus KQ_1 \oplus ... \oplus KQ_l \oplus ... .$$

KQ is an associative algebra. It has an identity if and only if Q_0 is finite and acyclic.

Let Q be a quiver. The two sided ideal of the path algebra KQ generated (as an ideal) by the arrows of Q is called the *arrow ideal* of KQ and is denoted by R_Q. So

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Ref.

[2] Ibrahim Assem, Daniel Simson and Andrzej Skowronski; *Elements of Representation Theory of Associative algebras, Vol.1 Techniques of Representation Theory*, London Mathematical Society Student Texts- vol. 65, Cambridge University press, 20.

$$R_Q = KQ_1 \oplus KQ_2 \oplus ... \oplus KQ_l \oplus ... .$$

Let R_Q^l denote the ideal of KQ generated, as a $K-$ vectorspace, by the set of all paths of length $\geq l$.

A two-sided ideal I of KQ is said to be *admissible* if there exists $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. If I is an admissible ideal of KQ, the pair (Q, I) is called *bound quiver* and the quotient algebra KQ/I is called a *bound quiver algebra*.

A Quiver Q is said to be *connected* if the underlying graph is connected. An algebra A is said to be *connected* if A is not a direct product of two algebras, or equivalently, 0 and 1 are the only central idempotents.

A partially ordered set X is said to be *locally finite* if, the subset $X_{yz} = \{x \in X : y \leq x \leq z\}$ is finite for each $y \leq z \in X$. The Incidence algebra $I(X, R)$ of a locally finite partially ordered set X over the commutative ring R with identity is $I(X, R) = \{f : X \times X \to R \mid f(x, y) = 0 \; if \; x \nleq y\}$
with operations defined by

$$(f + g)(x, y) = f(x, y) + g(x, y),$$
$$(f.g)(x, y) = \sum_{x \leq z \leq y} f(x, z).g(z, y),$$
$$(r.f)(x, y) = r.f(x, y)$$

for all $f, g \in I(X, R)$, $r \in R$ and $x, y, z \in X$.

The identity element of $I(X, R)$ is $\delta(x, y) = \begin{cases} 1 & if \; x = y \\ 0 & Otherwise \end{cases}$

For a finite partially ordered set X, the incidence algebra $I(X, K)$ is a subalgebra of the matrix algebra $M_n(K)$. The following theorem characterize finite dimensional incidence algebras.([1], Theorem 4.2.10)

Theorem 1. *Let K be a field and S be a subalgebra of $M_n(K)$. Then there exists a partially ordered set X of order n such that $I(X, K) \cong S$ if and only if*

(i) S contains n pairwise orthogonal idempotent and

(ii) $S/J(S)$ is commutative.

And, for incidence algebras of lower finite partially ordered sets we have the following characterization: ([3], Theorem 2.)

Theorem 2. *Let V be a $K-$vector space with dimension $|X|$, for a suitable set X. Let S be a subalgebra of $End_K V$. Then there exists a lower finite partial ordering in X such that $S \cong I(X, K)$ if and only if,*

Ref.

[1] E.Spiegel and C.J.O'Donnell ; *Incidence Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, Vol.206, Marcel Dekker, Newyork, 1997.

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(1) $1 \in S$

(2) $S/J(S)$ *is commutative*

(3) For each $x \in X$*, there is an* $E_x \in S$ *of rank 1, such that*

$$E_x.E_y = \delta_{xy}E_x \text{ and } \bigoplus_{x\in X} E_x(V) = V$$

(4) $X_y = \{z \in X \mid E_z.S.E_y \neq 0\}$ *is finite for each* $y \in X$

Notes

II. The Partially Ordered Set Corresponding To An Acyclic Quiver

Let Q be an acyclic quiver. Let Q_0 denote the set of all points of Q. We may define an order on Q_0 by $i \leq j$ if and only if there exists a path from i to j. Since $\varepsilon_a \in Q$, $\forall a \in Q_0$, we have $i \leq i$, $\forall i \in Q_0$. If $i \neq j$ and $i < j$, then $j \nleq i$, since Q is acyclic. If there exist a path α from i to j and β from j to k, $\alpha\beta$ is a path from i to k. So $i \leq j$ and $j \leq k$ implies $i \leq k$. So $(Q_0, \leq)$ is a partially ordered set. Clearly $(Q_0, \leq)$ is locally finite for a finite quiver Q.

Proposition 1. Let Q be a finite acyclic quiver such that there exists at most 1 path from i to j, for each pair $i, j \in Q_0$. Then the path algebra KQ is isomorphic to the incidence algebra $I(Q_0, K)$.

Proof. Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver such that, there exists at most 1 path from i to j, for each pair $i, j \in Q_0$. If $i \leq j$, denote the unique path from i to j by α_{ij}. Define $\phi : KQ \rightarrow I(Q_0, K)$ such that $\alpha_{ij} \mapsto E_{ij}$ where $E\delta_{ij}$ is the function which assumes the value 1 at (i, j) and zero elsewhere. This is an isomorphism from KQ to $I(Q_0, K)$, since ϕ is a bijective map from basis of KQ $(\{\alpha_{ij}|i, j \in Q_0\})$ to a basis of $I(Q_0, K)$ $(\{\delta_{ij}|i, j \in Q_0\})$ and it preserves addition, multiplication and identity element. Hence the theorem.

Definition 1. If a quiver $Q = (Q_0, Q_1)$ is such that there exists atmost one path from x to y for each pair $x, y \in Q_0$, then we call Q a *unique path quiver*.

Proposition 2. Let K be a field and S be a subalgebra of $M_n(K)$. Then there is a unique path quiver $Q = (Q_0, Q_1)$ with n vertices such that $KQ \cong S$ if and only if

(i) S contains n pairwise orthogonal idempotents and

(ii) $S/J(S)$ is commutative.

Proposition 3. Given a finite acyclic quiver $Q = (Q_0, Q_1)$ there exists a surjective homomorphism from KQ onto the associated incidence algebra $I(Q_0, K)$ and this becomes an isomorphism if and only if Q is such that, there exists atmost one path from i to j, for each pair $i, j \in Q_0$.

Proof. Since Q is finite and acyclic, KQ is finite dimensional with the set of all paths as its basis. Let $Q_0 = \{x_1, x_2, ..., x_n\}$ and let for each $i, j \in Q_0$, $i \leq j$ whenever there exists a path from i to j. $I(Q_0, K)$ will be isomorphic to a subalgebra S of $T_n(K)$. If E_{ij} denote the $n \times n$ matrix with 1 at the $(i,j)th$ position and zeros elsewhere. It is clear that whenever $i \leq j$, $E_{ij} \in S$. Now define $\varphi : KQ \to I(Q_0, K)$ such that $\varphi(\alpha) = E_{ij}$ if α is a path from i to j. If $\alpha : i \to j$ and $\beta : m \to n$ are two paths Q, $\alpha\beta = 0$ if $j \neq m$ and $\alpha\beta$ is a path from i to n, if $j = m$.

$$\varphi(\alpha\beta) = \begin{cases} E_{in} & \text{if } j = m \\ 0 & \text{Otherwise} \end{cases}$$

$$= E_{ij}.E_{mn}$$

$$= \varphi(\alpha).\varphi(\beta)$$

$\phi(\sum_{i \in Q_0} \varepsilon_i) = I_n$ since $\varepsilon_i \mapsto E_{ii}$. Hence ϕ is a surjective map from the basis of KQ onto a basis of $I(Q_0, K)$, which is compatible with the addition, multiplication and scalar multiplication. Hence ϕ is a surjective homomorphism from KQ onto $I(Q_0, K)$. Clearly if there exists at most one path from i to j for each pair $i, j \in Q_0$, then $dim(KQ) = dim(I(Q_0, K))$. So $KQ \cong I(Q_0, K)$.

Remark 1. Under the above defined surjective homomorphism ϕ, we can reach at the following results.

(1) If Q is a finite acyclic quiver then the Jacobson radical of KQ will be mapped on to the Jacobson Radical of $I(Q_0, K)$

(2) R_Q^l will be mapped on to the two sided ideal J_l of $I(Q_0, K)$, where $J_l = \{f \in I(Q_0, K) | f(x, y) = 0 \; if \; the \; length \; of \; the \; longest \; chain \; from \; x \; to \; y \; is \; \leq l\}$

Definition 2. Let $Q = (Q_0, Q_1)$ be an acyclic quiver. Then Q is said to be a *locally finite quiver*, if for each pair $i, j \in Q_0$, there exists only finitely many paths from i to j and is said to be *lower finite* if for each $x \in Q_0$ there exist only finitely many paths that ends at x.

Note that if $Q = (Q_0, Q_1)$ is an acyclic locally finite quiver, then the associated partial order set is also locally finite.

Proposition 4. If Q is an acyclic locally finite quiver and $(Q_0, \leq)$ is the associated locally finite partially ordered set, then there exists a homomorphism $\phi : KQ \to I(Q_0, K)$ and this homomorphism is injective if and only if Q is such that, for each pair $i, j \in Q_0$ there exists at most one path from i to j.

Notes

Proof. Let V be a $K-$vector space of dimension $\mid Q_0 \mid$. Let $\{v_i \mid i \in Q_0\}$ be a basis of V. For each pair $i, j \in Q_0$ there exists $E_{ij} \in End_K V$ such that $E_{ij}(v_k) = \delta_{jk} v_i$. Let $S = span\{E_{ij} \mid i, j \in Q_0, i \leq j\}$. This is a subalgebra of $I(Q_0, K)$, since E_{ij} can be mapped to $\delta_{ij} \in I(Q_0, K)$. These δ_{ij}s will span a subalgebra of $I(Q_0, K)$. Denote this subalgebra by A. We have, $S \cong A$. Call this isomorphism by ψ .

Notes

Now, consider a basis of KQ, which is the set of all paths in Q. If α is a path from i to j, then define $\phi : KQ \to S$ such that $\alpha \mapsto E_{ij}$.This is a homomorphism from KQ to S.

Now, $\phi \circ \psi : KQ \to I(Q_0, K)$ is a homomorphism. It is clear that this becomes injective if and only if there exists at most one path from i to j for each pair $i, j \in Q_0$.

Remark 2. KQ has an identity if and only if Q is finite and acyclic. But $I(Q_0, K)$ always has an identity. So that $\phi \circ \psi$ can not be surjective in general.

Remark 3. Associated to a finite acyclic quiver we get a unique partially ordered set. But the converse is not true. For example, corresponding to $X = \{1, 2\}$ together with the usual ordering we get countably many quivers with n arrows between 1 and 2 for any natural number $n \in \mathbb{N}$.

III. Path Algebra: A Generalized Definition

Definition 3. Let Q be a quiver, and let P be the set of all paths in Q. A *Path Algebra* of Q is defined as $\left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha \in K, \alpha \in P \right\}$. We define addition and scalar multiplication componentwise. If $(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b)$ and $(c \mid \beta_1, \beta_2, ..., \beta_m \mid d)$ are any paths in Q, we define their product as,

$(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b).(c \mid \beta_1, \beta_2, ..., \beta_m \mid d) = \delta_{bc}(a \mid \alpha_1, ..., \alpha_l, \beta_1, ...\beta_m \mid d)$. The product of two arbitrary elements of KQ can be defined by assuming distributivity of multiplication of paths over arbitrary summation.

$$\therefore \left(\sum_{\alpha \in P} c_\alpha \alpha \right) \left(\sum_{\beta \in P} d_\beta \beta \right) = \sum_{\alpha, \beta \in P} c_\alpha d_\beta \alpha \beta$$

This is well defined since $\alpha\beta = 0$ if $t(\alpha) \neq s(\beta)$ and since $\alpha\beta$ is a path, it is of finite length and so it can be expressed as a product of 2 paths only in finitely many ways.

$$\text{Define } KQ_l = \left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha = 0 \; if \; length \; of \; \alpha \neq l \right\}.$$

KQ can be expressed as a direct product of KQ_l for $l \geq 0$. i.e.,

$$KQ = KQ_0 \times KQ_1 \times ... \times KQ_l \times ...$$

Clearly $(KQ_n).(KQ_m) \subseteq KQ_{n+m}$ $\forall n, m \geq 0$.

Note that if Q is a finite acyclic quiver, then our generalized definition and old definition of path algebra coincides. So the results we obtained in the previous section for finite acyclic quiver holds, even when we use the generalized definition of path algebra. For a finite acyclic quiver Q, the set of all its paths P, will serve as a basis for KQ. Here after we use the generalized definition of path algebra.

Notes

Proposition 5. Let Q be a quiver and KQ be the corresponding path algebra.Then,

(a) KQ is an associative algebra.

(b) The element $\sum\limits_{a \in Q_0} \varepsilon_a$ is the identity in KQ.

(c) KQ is finite dimensional if and only if Q is finite and acyclic.

Proof. (a) The fact that KQ is an associative algebra, follows directly from the definition of multiplication, because, the product of paths is the composition of paths and hence it is associative. Any element in KQ is an arbitrary linear combination of paths. So associativity holds in general, since we have distributivity of multiplication over arbitrary summation.

(b) Let $\sum\limits_{\alpha \in P} c_\alpha \alpha \in KQ$ be arbitrary.

$$\left(\sum_{a \in Q_0} \varepsilon_a\right) \cdot \left(\sum_{\alpha \in P} c_\alpha \alpha\right) = \sum_{\alpha \in P} c_\alpha \left[\left(\sum_{a \in Q_0} \varepsilon_a\right) .\alpha\right]$$
$$= \sum_{\alpha \in P} c_\alpha \left(\sum_{a \in Q_0} \varepsilon_a .\alpha\right),$$
$$= \sum_{\alpha \in P} c_\alpha \alpha$$

since $\varepsilon_a.\alpha = \begin{cases} \alpha, \text{ if } s(\alpha) = a \\ 0, \text{otherwise} \end{cases}$

Similarly since $\alpha.\varepsilon_a = \begin{cases} \alpha, \text{ if } t(\alpha) = a \\ 0, \text{otherwise} \end{cases}$,

we get $\left(\sum\limits_{\alpha \in P} c_\alpha \alpha\right) . \left(\sum\limits_{a \in Q_0} \varepsilon_a\right) = \left(\sum\limits_{\alpha \in P} c_\alpha \alpha\right)$

Therefore, $\sum\limits_{a \in Q_0} \varepsilon_a$ serves as the identity of KQ.

(*c*) If Q is infinite, so is the set P. Span$(P) \subseteq KQ$ and P is linearly independent. So that KQ is infinite dimensional.

Now if Q is cyclic, then there is atleast one cycle, say ω in Q.
Then $\omega^l \in P \, \forall \, l \geq 1$, which implies P is infinite and hence KQ is also infinite dimensional.

Conversely, if Q is finite and acyclic, then $\mid P \mid$ is finite and in this case P serves as a basis for KQ. Hence KQ is finite dimensional.

Notes

Proposition 6. Let $Q = (Q_0, Q_1)$ be a unique path quiver then an element $a \in KQ$ is a unit if and only if the coefficient a_{xx} of the stationary path ε_x is nonzero for all $x \in Q_0$.

Proof. Let $a = \sum\limits_{x,y \in Q_0} a_{xy}\alpha_{xy}$ be a unit element of KQ, where α_{xy} is the unique path from x to y, if there is one. Then there exists a $b = \sum\limits_{x,y \in Q_0} b_{xy}\alpha_{xy}$ in KQ such that $ab = \sum\limits_{x \in Q_0} \varepsilon_x$. That is

$$\sum_{x,y,z,u \in Q_0} a_{xy} b_{zu} \alpha_{xy} \alpha_{zu} = \sum_{x \in Q_0} \varepsilon_x$$

$$\Rightarrow \sum_{x,u \in Q_0} \left. \sum_{y \in Q_0} a_{xy} b_{yu} \right) \alpha_{xu} = \sum_{x \in Q_0} \varepsilon_x$$

Equating coefficients on both sides we may conclude that the coefficients of each stationary path should be nonzero.

Conversely, suppose that $a = \sum\limits_{x,y \in Q_0} a_{xy}\alpha_{xy}$ is such that $a_{xx} \neq 0$ for all $x \in Q_0$. Then there is an element $b \in KQ$ such that

$$b_{xy} = 1/a_{xx}, \; if \;\; x = y$$

$$= \tfrac{-1}{a_{xx}} \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy}, \; if \;\; x \neq y$$

So that if $x = y$ coefficient of $\varepsilon_x = a_{xx}.b_{xx} = 1$ and

if $x \neq y$, coefficient of α_{xy} in the product $a.b = \sum\limits_{z \in Q_0} a_{xz} b_{zy}$

But,

$$\begin{aligned} \sum_{z \in Q_0} a_{xz} b_{zy} &= \quad a_{xx} b_{xy} + \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy} \\ &= a_{xx} \tfrac{-1}{a_{xx}} \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy} + \sum_{z \in Q_0 - \{x\}} a_{xz} b_{zy} \\ &= 0 \end{aligned}$$

Hence $a.b = \sum\limits_{x \in Q_0} \varepsilon_x$ which implies that a is a unit.

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Remark 4. $\{\varepsilon_a \mid a \in Q_0\}$ of all stationary paths in Q is a set of primitive orthogonal idempotents for KQ such that $\sum\limits_{a \in Q_0} \varepsilon_a = 1 \in KQ$.

Proposition 7. Let Q be a quiver and KQ be its path algebra. Then KQ is connected if and only if Q is connected.

Proof. To prove this, we first prove that KQ is connected if and only if there does not exist a nontrivial partition $I \dot{\cup} J$ of Q_0 such that if $i \in I$ and $j \in J$ then, $\varepsilon_i(KQ)\varepsilon_j = 0 = \varepsilon_j(KQ)\varepsilon_i$. Assume that there exists such a partition for Q_0. Let $c = \sum\limits_{j \in J} \varepsilon_j$. Since the partition is nontrivial $c \neq 0$ or 1. Since ε_j's are primitive orthogonal idempotents and multiplication in KQ is distributive over arbitrary sum, we can conclude that c is an idempotent. Also,

$c.\varepsilon_i = 0 = \varepsilon_i.c, \ \forall i \in I$ and

$c.\varepsilon_j = 0 = \varepsilon_j.c, \ \forall j \in J.$

According to our hypothesis $\varepsilon_i.a.\varepsilon_j = 0 = \varepsilon_j.a.\varepsilon_i$, $\forall i \in I$ and $\forall j \in J$ and $\forall a \in KQ$.

Therefore,

$$\begin{aligned} c.a &= \left. \sum_{j \in J} \varepsilon_j \right) .a \\ &= \left. \sum_{j \in J} \varepsilon_j.a \right) .1 \\ &= \left. \sum_{j \in J} \varepsilon_j.a \right) . \left(\sum_{i \in I} \varepsilon_i + \sum_{k \in J} \varepsilon_k \right) \\ &= \sum_{k,j \in J} \varepsilon_j a \varepsilon_k \\ &= \left. \sum_{j \in J} \varepsilon_j + \sum_{i \in I} \varepsilon_i \right) a \left(\sum_{k \in J} \varepsilon_k \right) \\ &= a.c \end{aligned}$$

which implies c is a nontrivial central idempotent. Hence KQ is not connected. Conversely, if KQ is not connected, it contains a nontrivial central idempotent, say c.

Therefore,

$$\begin{aligned} c &= 1.c.1 \\ &= \left. \sum_{i \in Q_0} \varepsilon_i \right) .c. \left. \sum_{j \in Q_0} \varepsilon_j \right) \\ &= \sum_{i,j \in Q_0} \varepsilon_i c \varepsilon_j \end{aligned}$$

Notes

$$= \sum_{i \in Q_0} \varepsilon_i c \varepsilon_i, \text{ since } c \text{ is central}$$

Now let $c_i = \varepsilon_i c = c\varepsilon_i = \varepsilon_i c \varepsilon_i \in \varepsilon_i (KQ)\varepsilon_i$

So that, $c_i^2 = (\varepsilon_i c \varepsilon_i)(\varepsilon_i c \varepsilon_i) = \varepsilon_i c^2 \varepsilon_i = \varepsilon_i c \varepsilon_i = c_i$,

hence c_i is an idempotent.

But ε_i's are primitive, so that either $c_i = 0$ or $c_i = 1$, since

Notes

$\varepsilon_i = \varepsilon_i (1 - c_i + c_i)$

$= \varepsilon_i (1 - c_i) + \varepsilon_i c_i$

So, $\varepsilon_i = \varepsilon_i c_i$ or $\varepsilon_i = \varepsilon_i (1 - c_i)$.

Let $I = \{i \in Q_0 / c_i = 0\}$ and $J = \{j \in Q_0 / c_j = 1\}$. Since $c \neq 0, 1$, this is a nontrivial partition of Q_0. And if $i \in I$ then, $\varepsilon_i c = c\varepsilon_i = 0$ and if $j \in J$ then, $\varepsilon_j c = c\varepsilon_j = \varepsilon_j$.

Therefore if $i \in I$ and $j \in J$, $\varepsilon_i(KQ)\varepsilon_j = \varepsilon_i(KQ)c\varepsilon_j = \varepsilon_i c(KQ)\varepsilon_j = 0$.

Similarly, $\varepsilon_j(KQ)\varepsilon_i = 0$.

Now assume that KQ is not connected. Let Q' be a connected component of Q. Let Q'' be the full subquiver of Q having the set of points $Q_0'' = Q_0 \backslash Q_0'$. Since Q is not connected, both Q_0' and Q_0'' are nonempty. Let $a \in Q_0'$ and $b \in Q_0''$. Since Q is not connected, then if α is any path in Q, either α is entirely contained in Q' or α is entirely contained in Q''

If α is contained in Q' then, $\alpha.\varepsilon_b = 0$ and so $\varepsilon_a.\alpha.\varepsilon_b = 0$.

If α is contained in Q'' then, $\varepsilon_a.\alpha = 0$ and so $\varepsilon_a.\alpha.\varepsilon_b = 0$.

Therefore, $\varepsilon_a (KQ) \varepsilon_b = 0$. Similarly, $\varepsilon_b (KQ) \varepsilon_a = 0$

This implies KQ is not connected.

Now assume that Q is connected but KQ is not. We have a nontrivial disjoint union of Q_0 such that $Q_0 = Q_0' \dot{\cup} Q_0''$ and if $a \in Q'$ and $b \in Q''$ then, $\varepsilon_a (KQ) \varepsilon_b = 0 = \varepsilon_b (KQ) \varepsilon_a$.

Since Q is connected, there exists some $a_0 \in Q_0'$ and some $b_0 \in Q_0''$ such that they are neighbors. Without loss of generality, suppose that there exists an arrow $\alpha : a_0 \to b_0$. Therefore, $\alpha = \varepsilon_{a_0}.\alpha.\varepsilon_{b_0} \in \varepsilon_{a_0} (KQ) \varepsilon_{b_0} = 0$, which is a contradiction. Hence KQ is connected.

Definition 4. Let Q be a quiver and KQ be its path algebra. The two-sided ideal of KQ, is called *arrow ideal* and is denoted by R_Q if it is defined by,

$$R_Q = \left\{ \sum_{\alpha \in P} c_\alpha \alpha | \ c_\alpha = 0, \text{ if } \alpha \text{ is a stationary path} \right\}$$

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Let R_Q^l denote the two-sided ideal of KQ generated by the paths of length $\geq l$. So that

$$R_Q^l = \left\{ \sum_{\alpha \in P} c_\alpha \alpha \mid c_\alpha = 0, \text{ if } \alpha \text{ is a path of length less than } l \right\}.$$

Therefore $\frac{R_Q^l}{R_Q^{l+1}} \cong KQ_l$

Definition 5. A two-sided ideal I of KQ is said to be *admissible* if there exists $m \geq 2$ such that

$$R_Q^m \subseteq I \subseteq R_Q^2.$$

If I is an admissible ideal of KQ, the pair (Q, I) is called *bound quiver* and the quotient algebra KQ/I is called a *bound quiver algebra*.

Proposition 8. Let Q be a quiver and I be an admissible ideal of KQ. The set $\{e_a = \varepsilon_a + I \mid a \in Q_0\}$ is a set of primitive orthogonal idempotents of the bound quiver algebra KQ/I and $\sum_{a \in Q_0} e_a = 1_{KQ/I}$

Proof. Since e_a is the image of ε_a under the canonical homomorphism from $KQ \to KQ/I$, and $\sum_{a \in Q_0} \varepsilon_a = 1$, it is clear that $\{e_a = \varepsilon_a + I \mid a \in Q_0\}$ is a set of orthogonal idempotents such that $\sum_{a \in Q_0} e_a = 1_{KQ/I}$. Now we have to prove that each e_a is primitive. That is only idempotents of $e_a (KQ/I) e_a$ are zero and e_a. Any idempotent of $e_a (KQ/I) e_a$ can be written in the form $e = \lambda \varepsilon_a + \omega + I$, $\lambda \in K$ and ω is a linear combination of cycles of length ≥ 1. Therefore, since e is an idempotent,

$$(\lambda \varepsilon_a + \omega)^2 + I = (\lambda \varepsilon_a + \omega) + I$$

$$i.e., \ (\lambda \varepsilon_a + \omega)^2 - (\lambda \varepsilon_a + \omega) \in I$$

$$i.e., \ (\lambda^2 - \lambda) \varepsilon_a + (2\lambda - 1) \omega + \omega^2 \in I$$

Since $I \subseteq R_Q^2$, $(\lambda^2 - \lambda) = 0$ which implies $\lambda = 0 \, or \, 1$

If $\lambda = 0$, $e = \omega + I$ and then, ω is an idempotent modulo I.Since $R_Q^m \subseteq I$ for some $m \geq 2$, $\omega^m \in I$ and so $\omega \in I$. So that $e = 0 \in KQ/I$.

If $\lambda = 1$, then $e = \varepsilon_a + \omega + I$ and $e_a - e = -\omega + I$ is an idempotent in $e_a (KQ/I) e_a$. So that ω is an idempotent modulo I, which implies $\omega^m \in I$ which in turn implies that $\omega \in I$. Hence $e_a - e \in I$ and $e_a = e$ modulo I.

Proposition 9. Let Q be a quiver and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is connected if and only if Q is a connected quiver.

Notes

Proof. Let Q be not connected. By Proposition 5, we have KQ is not connected. And this implies that there exists a nontrivial central idempotent γ (*neither* 0 *nor* 1) which can be chosen as a sum of paths of stationary paths. Then $c = \gamma + I \neq I$. If $c = 1 + I$ then, $1 - \gamma \in I$, which is not possible, since $I \subseteq R_Q^2$. Hence c is a nontrivial central idempotent of KQ/I and so KQ/I is not connected as an algebra.

Notes

Conversely, assume that Q is a connected quiver, but KQ/I is not a connected algebra. Then, there exists a nontrivial partition $Q_0 = Q_0' \dot{\cup} Q_0''$ such that whenever $x \in Q_0'$ and $y \in Q_0''$, then $e_x (KQ/I) e_y = 0 = e_y (KQ/I) e_x$. Since Q is a connected quiver, There is some $a \in Q_0'$ and $b \in Q_0''$ that are neighbors. With out loss of generality we may assume that there exists an arrow from a to b. Then, $\alpha = \varepsilon_a \alpha \varepsilon_b$ and so, $\overline{\alpha} = \alpha + I$ satisfies $\overline{\alpha} = e_a \overline{\alpha} e_b \in e_a (KQ/I) e_b = 0$. As $\overline{\alpha} \neq I$ $\left(\because I \subseteq R_Q^2\right)$, This is a contradiction. Hence KQ/I is connected.

IV. The Relation Between The Path Algebra of an Acyclic Quiver and the Incidence Algebra of the Associated Poset

In the second section, we discussed some homomorphism between Path algebras of finite and acyclic quivers and Incidence algebras of associated partially ordered sets. Now we discuss the same for infinite dimensional algebras.

Proposition 10. Let Q be a unique path quiver. Then $KQ \cong I(Q_0, K)$

Proof. Given that there exists atmost one path from x to y for each pair $x, y \in Q_0$. Denote this path by α_{xy}. An arbitrary element $a \in KQ$ can be written as

$$a = \sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy}$$

Define $\Phi : KQ \to I(Q_0, K)$ by $\Phi(a) = f_a$ where,

$$f_a(x, y) = a_{xy}$$

If $x \nleq y$, there is no path from x to y, so that the coefficient of α_{xy} in $a = a_{xy} = 0$.

So that $f_a(x, y) = 0$. Hence $f_a \in I(Q_0, K)$.

Now let $a = \sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy}$ and $b = \sum_{\alpha_{xy} \in P} b_{xy} \alpha_{xy}$

Then,

$$f_{a+b} = \Phi(a + b) = \Phi\left(\sum_{\alpha_{xy} \in P} a_{xy} \alpha_{xy} + \sum_{\alpha_{xy} \in P} b_{xy} \alpha_{xy}\right)$$

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$$= \Phi\left(\sum_{\alpha_{xy}\in P}(a_{xy}+b_{xy})\alpha_{xy}\right)$$

So that $\Phi(a+b) = \Phi(a) + \Phi(b)$.

Let $\Phi(ab) = f_{ab}$. Then,

$$\begin{aligned}\Phi(ab) &= \Phi\left(\left(\sum_{\alpha_{xy}\in P} a_{xy}\alpha_{xy}\right)\sum_{\alpha_{uv}\in P} b_{uv}\alpha_{uv}\right)\right)\\ &= \Phi\left(\sum_{\alpha_{xy},\alpha_{uv}\in P} a_{xy}b_{uv}\left(\alpha_{xy}\alpha_{uv}\right)\right)\\ &= \Phi\sum_{\alpha_{xv}\in P}\sum_{x\leqslant y\leqslant v} a_{xy}b_{yv}\right)\alpha_{xv}\right)\end{aligned}$$

Therefore, $f_{ab}(x,y) = \sum_{x\leqslant z\leqslant y} a_{xz}b_{zy}\right) = (f_a.f_b)(x,y)$, which implies $\Phi(ab) = \Phi(a).\Phi(b)$

$$\Phi\sum_{a\in Q_0}\varepsilon_a\right) = \delta = \text{identity in } I(Q_0,K)$$

$\Phi(c.a) = c.\Phi(a)$, for $c\in K$, $a\in KQ$

So that Φ is a homomorphism from KQ to $I(Q_0,K)$.

Now let $f\in I(Q_0,K)$, then there exists an $a = \sum_{\alpha_{xy}\in P} f(x,y)\alpha_{xy}\in KQ$ such that $\Phi(a) = f$. Hence Φ is onto.

If $a,b\in KQ$ *such that* $\Phi(a)=\Phi(b)$ then,

$$\begin{aligned}&\Phi(a)(x,y) = \Phi(b)(x,y)\ \ \forall x,y\in Q_0\\ &i.e.\quad a_{xy} = b_{xy}\ \ \forall x,y\in Q_0\\ &i.e.\quad a = b\end{aligned}$$

So Φ is one-one and hence it is an isomorphism.

Combining theorem 2 and proposition 10 we can reach at the following result

Proposition 11. Let K be a field and V be a $K-$vectorspace. Let S be a subalgebra of $End_K(V)$. Then there exists a lower finite unique path quiver $Q=(Q_0,Q_1)$ with $|Q_0| = dim(V)$ such that $KQ\cong S$ if and only if

(i) $1\in S$

Notes

(ii) $S/J(S)$ is commutative.

(iii) For each $x \in Q_0$, there is $E_x \in S$ of rank 1 such that $E_x.E_y = \delta_{xy}E_x$ where δ_{xy} is the Kronecker's delta and $\sum_{x \in Q_o} E_x(V) = V$

(iv) $X_y = \{z \in Q_0 : E_z.S.E_y \neq 0\}$ is finite for each $y \in Q_0$

Notes

Proposition 12. Let Q be a locally finite acyclic quiver. Then there exists a surjective homomorphism from KQ to $I(Q_0, K)$.

Proof. Let Q be a locally finite acyclic quiver and P be the set of all paths in Q. Since Q is locally finite, there exists only finitely many paths from x to y for each pair $x, y \in Q_0$. Let n_{xy} denote the number of paths from x to y in Q, and let $\alpha_{xy}^{(1)}, \alpha_{xy}^{(2)}, ..., \alpha_{xy}^{(n_{xy})}$ denote the n_{xy} paths from x to y in Q. Let $a \in KQ$ be arbitrary. So that a can be written as $a = \sum_{\alpha \in P} a_\alpha \alpha$. Let a_{xy} denote the sum of coefficients of all paths from x to y that comes in a. Define $\Phi : KQ \to I(Q_0, K)$ by $\Phi(a) = f_a$, where $f_a(x, y) = a_{xy}$ As in the previous proposition, it is easy to verify that $f_a \in I(Q_0, K)$, Φ preserves addition and scalar multiplication, Φ maps identity of KQ to identity of $I(Q_0, K)$. Now we prove that Φ preserves multiplication. Let $a = \sum_{\alpha \in P} a_\alpha \alpha$ and $b = \sum_{\beta \in P} b_\beta \beta$. So that $ab = \sum_{\alpha, \beta \in P} a_\alpha b_\beta \alpha\beta$. Let us denote the sum of coefficients of all paths from x to y that comes in ab by $(ab)_{xy}$. Note that $\alpha\beta$ is a path from x to y if and only if $s(\alpha) = x$ and $t(\beta) = y$ and $t(\alpha) = s(\beta)$. So,

$$\begin{aligned}
(ab)_{xy} &= \sum_{\substack{x \leq z \leq y \\ 1 \leq m \leq n_{xz} \\ 1 \leq n \leq n_{zy}}} a_{\alpha_{xz}^{(m)}} b_{\beta_{zy}^{(n)}} \\
&= \sum_{x \leq z \leq y} \left(\sum_{1 \leq m \leq n_{xz}} a_{\alpha_{xz}^{(m)}} \right) \left(\sum_{1 \leq n \leq n_{zy}} b_{\beta_{zy}^{(n)}} \right) \\
&= \sum_{x \leq z \leq y} a_{xz} b_{zy} \\
&= \sum_{x \leq z \leq y} f_a(x, z) f_b(z, y) \\
&= (f_a.f_b)(x, y)
\end{aligned}$$

So that Φ is a homomorphism from KQ to $I(Q_0, K)$. Now, let $f \in I(Q_0, K)$ and denote any fixed path from x to y by α_{xy}. So that there exists some $a = \sum_{x, y \in Q_0} f(x, y)\alpha_{xy} \in KQ$ such that $\Phi(a) = f$.

Hence Φ is a surjective homomorphism.

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